Borel reducibility and Hölder(α) embeddability between Banach spaces

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Abstract

We investigate Borel reducibility between equivalence relations $E(X,p) = X^{\mathbb{N}}/\ell_p(X)$'s where X is a separable Banach space. We show that this reducibility is related to the so called Hölder(α) embeddability between Banach spaces. By using the notions of type and cotype of Banach spaces, we present many results on reducibility and unreducibility between $E(L_r,p)$'s and $E(c_0,p)$'s for $r,p \in [1,+\infty)$.

We also answer a problem presented by Kanovei in the affirmative by showing that $C(\mathbb{R}^+)/C_0(\mathbb{R}^+)$ is Borel bireducible to $\mathbb{R}^{\mathbb{N}}/c_0$.

Key words: Borel reducibility, Hölder(α) embeddability, type, cotype
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1. Introduction

Borel reducibility hierarchy of equivalence relations on Polish spaces becomes the main focus of invariant descriptive set theory. Some important equivalence relations, for example, $\mathbb{R}^{\mathbb{N}}/\ell_p$ and $\mathbb{R}^{\mathbb{N}}/c_0$, were investigated by R. Dougherty and G. Hjorth. They showed that, for $p,q \in [1,+\infty)$,

$$\mathbb{R}^{\mathbb{N}}/\ell_p \leq_B \mathbb{R}^{\mathbb{N}}/\ell_q \iff p \leq q,$$

while $\mathbb{R}^{\mathbb{N}}/c_0$ and $\mathbb{R}^{\mathbb{N}}/\ell_p$ are Borel incomparable (see [5] and [10]).

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Let X be a topological linear space and Y a Borel linear subspace of X, then X/Y is a natural example of Borel equivalence relation. We are interested in the Borel reducibility between this kind of equivalence relations. One of the motivation for this paper is a problem asked by Kanovei that whether $C(\mathbb{R}^+)/C_0(\mathbb{R}^+) \sim_B \mathbb{R}/c_0$.

In this paper, we generalize Doutherty and Hjorth's results by considering equivalence relations $X^{\mathbb{N}}/\ell_p(X)$'s, which will be denoted by $E(X,p)$ where X is a separable Banach space and $p \in [1,+\infty)$. We show that Borel reducibility between this kind of equivalence relations is related to the existence of Hölder(α) embeddings.

Theorem 1.1. *Let X, Y be two separable Banach spaces, $p, q \in [1,+\infty)$. If there exists a Hölder($\frac{p}{q}$) embedding from X to $\ell_q(Y)$, then we have $E(X,p) \leq_B E(Y,q)$.*

On the other hand, via introducing a similar notion of finitely Hölder(α) embeddability, we prove the following theorem.

Theorem 1.2. *Let X, Y be two separable Banach spaces, $p, q \in [1,+\infty)$. If $E(X,p) \leq_B E(Y,q)$, then X finitely Hölder($\frac{p}{q}$) embeds into $\ell_q(Y)$.*

For investigating the notion of finitely Hölder(α) embaddability, we pay attention to the famous type-cotype theory in Banach space theory.

Theorem 1.3. *Let X, U be two infinite dimensional Banach spaces, $\alpha > 0$. If X finitely Hölder(α) embeds into U, then we have*

(1) $\alpha \leq 1$;
(2) $\frac{p(X)}{p(U)} \geq \alpha$;
(3) $p(U) > 1 \Rightarrow q(X) \leq q(U)$.

We apply this theorem to classical Banach spaces to show many results on reducibility and unreducibility between $E(L_r,p)$'s and $E(c_0,p)$'s. For instance, we show that

Theorem 1.4. *For $r, s \in [1,2]$ and $p, q \in [1,+\infty)$, if $s \leq q$, then*

$$E(L_r,p) \leq_B E(L_s,q) \iff p \leq q,\ \frac{r}{p} \geq \frac{s}{q}.$$

Equivalence relations $E(X,0) = X^{\mathbb{N}}/c_0(X)$'s are also considered. We answer Kanovei's problem in the affirmative by showing that

Theorem 1.5. *Let X be a separable Banach space, $p \in [1,+\infty)$. Then $E(X,0) \sim_B \mathbb{R}^{\mathbb{N}}/c_0$, and $E(X,p)$ is not Borel comparable with $\mathbb{R}^{\mathbb{N}}/c_0$.*

Since the proofs of several results of this paper are cited from [5], we shall assume that the reader has a copy of [5] handy.

The paper is organized as follows. In section 2 we recall some notions in descriptive set theory and functional analysis. In section 3 we answer Kanovei's problem. In section 4 we introduce the notion of finitely Hölder(α) embeddability and prove Theorem 1.1 and 1.2. In section 5 we prove Theorem 1.3. In section 6 we apply theorems from earlier sections to classical Banach spaces, and derive several interesting corollaries. Finally section 7 contains some further remarks.

2. Basic notation

For a set I, we denote by $|I|$ the cardinal of I.

A topological space is called a *Polish space* if it is separable and completely metrizable. Let X, Y be Polish spaces and E, F equivalence relations on X, Y respectively. A *Borel reduction* from E to F is a Borel function $\theta : X \to Y$ such that

$$(x,y) \in E \iff (\theta(x), \theta(y)) \in F$$

for all $x, y \in X$. We say that E is *Borel reducible* to F, denoted $E \le_B F$, if there is a Borel reduction from E to F. If $E \le_B F$ and $F \le_B E$, we say that E and F are *Borel bireducible* and denote $E \sim_B F$. Similarly, we say that E is *strictly Borel reducible* to F, denoted $E <_B F$, if $E \le_B F$ but not $F \le_B E$. We refer to [2], [8] and [11] for background on Borel reducibility.

For two metric spaces $(M,d), (M',d')$ and $\alpha > 0$. We say that M *Hölder(α) embeds* into M' if there exist $T : M \to M'$, $A > 0$ such that

$$\frac{1}{A} d(u,v)^\alpha \le d'(T(u),T(v)) \le A d(u,v)^\alpha$$

for $u, v \in M$.

As usual, we denote $L_p[0,1]$ by L_p for $p \in [1, +\infty)$ for the sake of brevity. For a Banach space X, we denote by $\ell_p(X)$ the Banach space whose underlying space is

$$\left\{ x \in X^{\mathbb{N}} : \sum_{n \in \mathbb{N}} \|x(n)\|_X^p < +\infty \right\},$$

with the norm

$$\|x\|_{X,p} = \left(\sum_{n \in \mathbb{N}} \|x(n)\|_X^p \right)^{\frac{1}{p}}.$$

For $m \in \mathbb{N}$, we also denote by $\ell_p^m(X)$ the finite dimensional space $(X^m, \|\cdot\|_{X,p})$ where $\|s\|_{X,p} = (\sum_{n=1}^m \|s(n)\|_X^p)^{\frac{1}{p}}$ for $s \in X^m$. Note that $\ell_p(\mathbb{R}) = \ell_p$ and $\ell_p^m(\mathbb{R}) = \ell_p^m$.

The notions of type and cotype of Banach spaces are powerful tools for investigating the local character of Banach spaces. An infinite dimensional Banach space X is said to have (Rademacher) *type* p for some $1 \leq p \leq 2$, if there is a constant $C < +\infty$ such that for every n and any sequence $(u_j)_{j=1}^n$ in X,

$$\left(\frac{1}{2^n} \sum_{\epsilon \in \{-1,1\}^n} \left\| \sum_{j=1}^n \epsilon_j u_j \right\|^p \right)^{\frac{1}{p}} \leq C \left(\sum_{j=1}^n \|u_j\|^p \right)^{\frac{1}{p}};$$

X is said to have (Rademacher) *cotype* q for some $q \geq 2$, if there is a constant $D > 0$ such that for every n and any sequence $(u_j)_{j=1}^n$ in X,

$$\left(\frac{1}{2^n} \sum_{\epsilon \in \{-1,1\}^n} \left\| \sum_{j=1}^n \epsilon_j u_j \right\|^q \right)^{\frac{1}{q}} \geq D \left(\sum_{j=1}^n \|u_j\|^q \right)^{\frac{1}{q}}.$$

The supremum of all types and the infimum of all cotypes of X are denoted by $p(X)$ and $q(X)$ respectively. For $p \in [1, +\infty)$, it is well known that

$$p(\ell_p) = p(L_p) = \min\{p, 2\}, \quad q(\ell_p) = q(L_p) = \max\{p, 2\}.$$

For any Banach space X, $r \in [1, +\infty)$, we have

$$p(\ell_r(X)) = \min\{p(X), r\}, \quad q(\ell_r(X)) = \max\{q(X), r\}.$$

(We can prove these formulas from Kahane's inequality, or see the remark in [16], pp.16) For more details on type and cotype, we refer to [12].

Let X be an infinite dimensional Banach space, $p \in [1, +\infty)$, we say that X contains ℓ_p^n's uniformly if for $n \in \mathbb{N}$ and $\varepsilon > 0$, there is a linear embedding $T_n : \ell_p^n \to X$ such that $\|T_n\| \cdot \|T_n^{-1}\| \leq 1 + \varepsilon$. The Maurey-Pisier Theorem [13] shows that X contains $\ell_{p(X)}^n$'s and $\ell_{q(X)}^n$'s uniformly.

3. Borel bireducibility with $\mathbb{R}^\mathbb{N}/c_0$

Kanovei's problem is the following question [7], Question 7.5. Another version of this problem is the Question 16.7.2 of [11].

We denote the space of all positive real numbers by $\mathbb{R}^+$.

Question 3.1 (Kanovei). *Let $C(\mathbb{R}^+)$ be the space of continuous functions on $\mathbb{R}^+$. We define an equivalence relation E_K by*

$$fE_Kg \iff \lim_{t\to+\infty}(f(t) - g(t)) = 0$$

for $f, g \in C(\mathbb{R}^+)$. Is $E_K \sim_B \mathbb{R}^\mathbb{N}/c_0$?

Before answering this question, we consider a class of equivalence relations similar to E_K.

Definition 3.2. *Let $(M_n, d_n), n \in \mathbb{N}$ be a sequence of separable complete metric spaces. We define an equivalence relation $E((M_n)_{n\in\mathbb{N}}, 0)$ on $\prod_{n\in\mathbb{N}} M_n$ by*

$$(x, y) \in E((M_n)_{n\in\mathbb{N}}, 0) \iff \lim_{n\to\infty} d_n(x(n), y(n)) = 0$$

for $x, y \in \prod_{n\in\mathbb{N}} M_n$. If $(M_n, d_n) = (M, d)$ for $n \in \mathbb{N}$, we write $E(M, 0) = E((M_n)_{n\in\mathbb{N}}, 0)$ for the sake of brevity.

This notion was firstly introduced by I. Farah [6], denoted $\mathrm{D}(\langle M_n, d_n\rangle)$. Many results on $\mathrm{D}(\langle M_n, d_n\rangle)$ were given in [6], especially the case named c_0-equalities that all sets M_n are finite. By this definition, we have $E(\mathbb{R}, 0) = \mathbb{R}^\mathbb{N}/c_0$. We can see that E_K is Borel reducible to $E(C[0, 1], 0)$ with the reducing map $\theta_0 : C(\mathbb{R}^+) \to C[0, 1]^\mathbb{N}$ defined as

$$\theta_0(f)(n)(t) = f(t + n + 1)$$

for $n \in \mathbb{N}$ and $t \in [0, 1]$. To answer Kanovei's problem in the affirmative, it will suffice to show that $E(C[0, 1], 0) \leq_B \mathbb{R}^\mathbb{N}/c_0$. In order to do so, we need a theorem of I. Aharoni.

Theorem 3.3 (Aharoni [1]). *There is a constant $K > 0$ such that for any separable metric space (M, d), there is a map $T : M \to c_0$ satisfying*

$$d(u, v) \leq \|T(u) - T(v)\|_{c_0} \leq Kd(u, v)$$

for every $u, v \in M$.

We denote $I_n = \left\{\frac{k}{2^n} : k \in \mathbb{N}, 0 \leq k \leq 2^n\right\}$.

Theorem 3.4. *Let $(M_n, d_n), n \in \mathbb{N}$ be a sequence of separable complete metric spaces, then we have*

(i) $E((M_n)_{n\in\mathbb{N}}, 0) \leq_B \mathbb{R}^{\mathbb{N}}/c_0$*;*
(ii) $E((M_n)_{n\in\mathbb{N}}, 0) \leq_B E([0, 1], 0)$*;*
(iii) $E((M_n)_{n\in\mathbb{N}}, 0) \leq_B E((I_n)_{n\in\mathbb{N}}, 0)$*.*

Proof. (i) By Aharoni's theorem, there are $K > 0$ and maps $T_n : M_n \to c_0$ such that

$$d_n(u, v) \leq \|T_n(u) - T_n(v)\|_{c_0} \leq Kd_n(u, v)$$

for $u, v \in M_n$. Fix a bijection $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$. We define $\theta : \prod_{n\in\mathbb{N}} M_n \to \mathbb{R}^{\mathbb{N}}$ by

$$\theta(x)(\langle n, m\rangle) = T_n(x(n))(m)$$

for $x \in \prod_{n\in\mathbb{N}} M_n$ and $n, m \in \mathbb{N}$. It is easy to see that θ is continuous.

Now we check that θ is a reduction.

For every $x, y \in \prod_{n\in\mathbb{N}} M_n$, if $(x, y) \in E((M_n)_{n\in\mathbb{N}}, 0)$, then

$$\lim_{n\to\infty} d_n(x(n), y(n)) \to 0.$$

So $\forall\varepsilon > 0 \exists N \forall n > N(d_n(x(n), y(n)) < \varepsilon)$. Since $\|T_n(x(n)) - T_n(y(n))\|_{c_0} \leq Kd_n(x(n), y(n)) < K\varepsilon$, we have

$$\forall n > N \forall m(|T_n(x(n))(m) - T_n(y(n))(m)| < K\varepsilon).$$

For $n \leq N$, since $T_n(x(n)), T_n(y(n)) \in c_0$, we have

$$\lim_{m\to\infty} |T_n(x(n))(m) - T_n(y(n))(m)| = 0.$$

Therefore, for all but finitely many (n, m)'s, we have

$$|\theta(x)(\langle n, m\rangle) - \theta(y)(\langle n, m\rangle)| = |T_n(x(n))(m) - T_n(y(n))(m)| < K\varepsilon.$$

Thus

$$\lim_{\langle n,m\rangle\to\infty} |\theta(x)(\langle n,m\rangle) - \theta(y)(\langle n,m\rangle)| = 0.$$

It follows that $\theta(x) - \theta(y) \in c_0$.

On the other hand, for every $x, y \in \prod_{n\in\mathbb{N}} M_n$, if $\theta(x) - \theta(y) \in c_0$, then

$$\forall\varepsilon > 0 \exists N \forall n > N \forall m (|\theta(x)(\langle n,m\rangle) - \theta(y)(\langle n,m\rangle)| < \varepsilon).$$

Therefore,

$$\begin{aligned} d_n(x(n), y(n)) &\leq \|T_n(x(n)) - T_n(y(n))\|_{c_0} \\ &= \sup_{m\in\mathbb{N}} |T_n(x(n))(m) - T_n(y(n))(m)| \\ &= \sup_{m\in\mathbb{N}} |\theta(x)(\langle n,m\rangle) - \theta(y)(\langle n,m\rangle)| \leq \varepsilon. \end{aligned}$$

It follows that $(x, y) \in E((M_n)_{n\in\mathbb{N}}, 0)$.

(ii) Denote

$$d'_n(u, v) = \frac{d_n(u, v)}{1 + d_n(u, v)}.$$

It is easy to see that

$$\lim_{n\to\infty} d_n(x(n), y(n)) = 0 \iff \lim_{n\to\infty} d'_n(x(n), y(n)) = 0.$$

So we may assume that $d_n(u, v) \leq 1$ for all $n \in \mathbb{N}$. By the same arguments in (i), there are $K > 0$, maps $T_n : M_n \to c_0$ such that

$$d_n(u, v) \leq \|T_n(u) - T_n(v)\|_{c_0} \leq K d_n(u, v) \leq K$$

for $u, v \in M_n$ and a reduction $\theta : \prod_{n\in\mathbb{N}} M_n \to \mathbb{R}^{\mathbb{N}}$ as $\theta(x)(\langle n,m\rangle) = T_n(x(n))(m)$ for $x \in \prod_{n\in\mathbb{N}} M_n$ and $n, m \in \mathbb{N}$.

There are real numbers $r_{n,m}$ $(n, m \in \mathbb{N})$ such that

$$T_n(u)(m) \in [r_{n,m}, r_{n,m} + K]$$

for $u \in M_n$. Now we can define a reducing map $\theta' : \prod_{n\in\mathbb{N}} M_n \to [0,1]^{\mathbb{N}}$ by

$$\theta'(x)(\langle n,m\rangle) = \frac{T_n(x(n))(m) - r_{n,m}}{K}$$

for $x \in \prod_{n\in\mathbb{N}} M_n$ and $n, m \in \mathbb{N}$.

(iii) For $x \in [0,1]^{\mathbb{N}}$ and $n \in \mathbb{N}$, denote

$$\vartheta(x)(n) = \frac{[x(n)2^n]}{2^n}.$$

Clearly ϑ is a Borel reducing map from $[0,1]^{\mathbb{N}} \to \prod_{n\in\mathbb{N}} I_n$. $\square$

Corollary 3.5. $E_K \sim_B \mathbb{R}^{\mathbb{N}}/c_0 \sim_B E([0,1],0) \sim_B E((I_n)_{n\in\mathbb{N}},0)$.

Proof. $\mathbb{R}^{\mathbb{N}}/c_0 \leq_B E_K$ is trivial. The remaining parts of the corollary follow from Theorem 3.4. $\square$

4. Borel reducibility between $E(X,p)$'s

R. Dougherty and G. Hjorth proved in [5] that, for $1 \leq p < q < +\infty$,

$$\mathbb{R}^{\mathbb{N}}/\ell^p <_B \mathbb{R}^{\mathbb{N}}/\ell^q.$$

In [10], it is also shown by G. Hjorth that, for $p \in [1,+\infty)$, $\mathbb{R}^{\mathbb{N}}/\ell^p$ and $\mathbb{R}^{\mathbb{N}}/c_0$ are $\leq_B$ incomparable.

In the same spirit as of $\mathbb{R}^{\mathbb{N}}/\ell^p$ and $E((M_n)_{n\in\mathbb{N}},0)$, we introduce the following equivalence relations.

Definition 4.1. *Let $(M_n,d_n), n \in \mathbb{N}$ be a sequence of separable complete metric spaces. For $p \in [1,+\infty)$, we define an equivalence relation $E((M_n)_{n\in\mathbb{N}},p)$ on $\prod_{n\in\mathbb{N}} M_n$ by*

$$(x,y) \in E((M_n)_{n\in\mathbb{N}},p) \iff \sum_{n\in\mathbb{N}} d_n(x(n),y(n))^p < +\infty$$

for $x,y \in \prod_{n\in\mathbb{N}} M_n$. If $(M_n,d_n) = (M,d)$ for every $n \in \mathbb{N}$, we write $E(M,p) = E((M_n)_{n\in\mathbb{N}},p)$ for the sake of brevity.

In this section, we mainly consider Borel reducibility between $E(X,p)$'s where X's are Banach spaces. It is straightforward to check that

$$E(X,p) = X^{\mathbb{N}}/\ell_p(X).$$

The following theorem gives a sufficient condition for $E((M_n)_{n\in\mathbb{N}},p) \leq_B E(X,q)$. This condition is quite complicated though we need only some special cases.

Theorem 4.2. *Let X be a separable Banach space, $p,q \in [1,+\infty)$, and let $(M_n,d_n), n \in \mathbb{N}$ be a sequence of separable complete metric spaces. If there are $A,C,D > 0$, a sequence of Borel maps $T_n : M_n \to \ell_q(X)$ and two sequences of non-negative real numbers $\varepsilon_n, \delta_n, n \in \mathbb{N}$ such that*

(1) $\sum_{n\in\mathbb{N}} \varepsilon_n^p < +\infty, \sum_{n\in\mathbb{N}} \delta_n^q < +\infty$;

(2) $d_n(u,v) < \varepsilon_n \Rightarrow \|T_n(u) - T_n(v)\|_{X,q} < \delta_n$*;*
(3) $d_n(u,v) > C \Rightarrow \|T_n(u) - T_n(v)\|_{X,q} > D$*;*
(4) $\varepsilon_n \leq d_n(u,v) \leq C \Rightarrow$

$$\frac{1}{A} d_n(u,v)^{\frac{p}{q}} \leq \|T_n(u) - T_n(v)\|_{X,q} \leq A d_n(u,v)^{\frac{p}{q}}.$$

Then we have

$$E((M_n)_{n\in\mathbb{N}}, p) \leq_B E(X,q).$$

Before proving Theorem 4.2, we present several easy corollaries of it.

Corollary 4.3. *Let X be a separable Banach space and (M,d) a separable complete metric space, $p,q \in [1,+\infty)$. If M Hölder($\frac{p}{q}$) embeds into $\ell_q(X)$, then we have $E(M,p) \leq_B E(X,q)$.*

Corollary 4.4. *Let X and Y be two Banach spaces, $p,q \in [1,+\infty)$. If there exist $A,c,d > 0$ and a Borel map $T : X \to \ell_q(Y)$ satisfying that*

(1) $\|u - v\|_X < c \Rightarrow \|T(u) - T(v)\|_{Y,q} < d$*;*
(2) $\|u - v\|_X \geq c \Rightarrow$

$$\frac{1}{A}\|u - v\|_X^{\frac{p}{q}} \leq \|T(u) - T(v)\|_{Y,q} \leq A\|u - v\|_X^{\frac{p}{q}}.$$

Then we have $E(X,p) \leq_B E(Y,q)$.

Proof. Denote $\varepsilon_n = 2^{-n}c, \delta_n = (2^{\frac{p}{q}})^{-n}d$, define $T_n : X \to \ell_q(Y)$ by

$$T_n(u) = (2^{\frac{p}{q}})^{-n} T(2^n u).$$

Then the result follows from Theorem 4.2. $\square$

Proof of Theorem 4.2. Fix a bijection $\langle\cdot,\cdot\rangle : \mathbb{N}^2 \to \mathbb{N}$. We define $\theta : \prod_{n\in\mathbb{N}} M_n \to X^{\mathbb{N}}$ by

$$\theta(x)(\langle n,m\rangle) = T_n(x(n))(m)$$

for $x \in \prod_{n\in\mathbb{N}} M_n$ and $n,m \in \mathbb{N}$. It is easy to see that θ is Borel. By the definition we have

$$\begin{array}{rl} & \sum_{n,m\in\mathbb{N}} \|\theta(x)(\langle n,m\rangle) - \theta(y)(\langle n,m\rangle)\|_X^q \\ = & \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \|T_n(x(n))(m) - T_n(y(n))(m)\|_X^q \\ = & \sum_{n\in\mathbb{N}} \|T_n(x(n)) - T_n(y(n))\|_{X,q}^q . \end{array}$$

For $x, y \in \prod_{n\in\mathbb{N}} M_n$, we split $\mathbb{N}$ into three sets

$$I_1 = \{n \in \mathbb{N} : d_n(x(n), y(n)) < \varepsilon_n\},$$

$$I_2 = \{n \in \mathbb{N} : d_n(x(n), y(n)) > C\},$$

$$I_3 = \{n \in \mathbb{N} : \varepsilon_n \le d_n(x(n), y(n)) \le C\}.$$

From (2) we have

$$\sum_{n\in I_1} d_n(x(n), y(n))^p < \sum_{n\in I_1} \varepsilon_n^p \le \sum_{n\in\mathbb{N}} \varepsilon_n^p < +\infty,$$

$$\sum_{n\in I_1} \|T_n(x(n)) - T_n(y(n))\|_{X,q}^q < \sum_{n\in I_1} \delta_n^q \le \sum_{n\in\mathbb{N}} \delta_n^q < +\infty;$$

from (3) we have

$$\sum_{n\in I_2} d_n(x(n), y(n))^p > C^p|I_2|,$$

$$\sum_{n\in I_2} \|T_n(x(n)) - T_n(y(n))\|_{X,q}^q > D^q|I_2|;$$

and from (4) we have

$$\frac{1}{A^q}\sum_{n\in I_3} d_n(x(n), y(n))^p \le \sum_{n\in I_3} \|T_n(x(n)) - T_n(y(n))\|_{X,q}^q \le A^q \sum_{n\in I_3} d_n(x(n), y(n))^p.$$

Therefore,

$$\begin{array}{ll} & (x, y) \in E((M_n)_{n\in\mathbb{N}}, p) \\ \iff & \sum_{n\in\mathbb{N}} d_n(x(n), y(n))^p < +\infty \\ \iff & |I_2| < \infty,\ \sum_{n\in I_3} d_n(x(n), y(n))^p < +\infty \\ \iff & |I_2| < \infty,\ \sum_{n\in I_3} \|T_n(x(n)) - T_n(y(n))\|_{X,q}^q < +\infty \\ \iff & \sum_{n\in\mathbb{N}} \|T_n(x(n)) - T_n(y(n))\|_{X,q}^q < +\infty \\ \iff & \sum_{n,m\in\mathbb{N}} \|\theta(x)(\langle n, m\rangle) - \theta(y)(\langle n, m\rangle)\|_X^q < +\infty \\ \iff & \theta(x) - \theta(y) \in \ell^q(X). \end{array}$$

It follows that $E((M_n)_{n\in\mathbb{N}}, p) \le_B E(X, q)$. □

For Borel reducibility between $E(X, p)$'s, we aim to present a necessary condition which will be named finitely Hölder($\frac{p}{q}$) embeddability. Now we focus on the equivalence relations $E((Z_n)_{n\in\mathbb{N}}, p)$ where $Z_n, n \in \mathbb{N}$ are a sequence of finite metric spaces.

The following lemma is due to R. Dougherty and G. Hjorth.

Lemma 4.5. *Let Y be a separable Banach space, $p \in \{0\} \cup [1, +\infty)$, $q \in [1, +\infty)$, and let $(Z_n, d_n), n \in \mathbb{N}$ be a sequence of finite metric space. Assume that $E((Z_n)_{n\in\mathbb{N}}, p) \leq_B E(Y, q)$. Then there exist strictly increasing sequences of natural numbers $(b_j)_{j\in\mathbb{N}}, (l_j)_{j\in\mathbb{N}}$ and $T_j : Z_{b_j} \to Y^{l_{j+1}-l_j}$ such that, for $x, y \in \prod_{j\in\mathbb{N}} Z_{b_j}$, we have*

$$(x, y) \in E((Z_{b_j})_{j\in\mathbb{N}}, p) \iff \sum_{j\in\mathbb{N}} \|T_j(x(j)) - T_j(y(j))\|_{Y,q}^q < +\infty.$$

Proof. The proof is, almost word for word, a copy of the proof of [5], Theorem 2.2, Claim (i)-(iii). $\square$

Let X, Y be two separable Banach spaces, $p, q \in [1, +\infty)$. Assume that $E(X, p) \leq_B E(Y, q)$.

Fix a sequence of finite subsets $F_n \subseteq X, n \in \mathbb{N}$ such that

$$\{0\} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

and $\bigcup_{n\in\mathbb{N}} F_n$ is dense in X. For every $n \in \mathbb{N}$, we denote

$$Z_n = \left\{u + \frac{i}{2^n}(v - u) : u, v \in F_n, 0 \leq i \leq 2^n\right\}.$$

Then $F_n \subseteq Z_n$.

Since $Z_n \subseteq X$ is a sequence of finite metric spaces, we can find $(b_j)_{j\in\mathbb{N}}, (l_j)_{j\in\mathbb{N}}$ and $T_j : Z_{b_j} \to Y^{l_{j+1}-l_j}$, as in Lemma 4.5. Then we have the following lemma.

Lemma 4.6. *There exists an $m \in \mathbb{N}$ such that $\forall k \exists N \forall j > N$, for $u, v \in F_{b_j}$, if $\frac{1}{k} \leq \|u - v\|_X \leq 1$, then we have*

$$\frac{1}{2^m}\|u - v\|_X^{\frac{p}{q}} \leq \|T_j(u) - T_j(v)\|_{Y,q} \leq 2^m\|u - v\|_X^{\frac{p}{q}}.$$

Proof. Assume for contradiction that, for every m, $\exists k_m \exists^\infty j \exists u_j, v_j \in F_{b_j}$ such that $\frac{1}{k_m} \leq \|u_j - v_j\|_X \leq 1$ but either

$$\frac{1}{2^m}\|u_j - v_j\|_X^{\frac{p}{q}} > \|T_j(u_j) - T_j(v_j)\|_{Y,q}$$

or

$$\|T_j(u_j) - T_j(v_j)\|_{Y,q} > 2^m\|u_j - v_j\|_X^{\frac{p}{q}}.$$

We define two subsets $I_1, I_2 \subseteq \mathbb{N}$. For $m \in \mathbb{N}$, we put $m \in I_1$ iff $\exists k_m \exists^\infty j \exists u_j, v_j \in F_{b_j}$ satisfying that $\frac{1}{k_m} \leq \|u_j - v_j\|_X \leq 1$ and

$$\frac{1}{2^m}\|u_j - v_j\|_X^{\frac{p}{q}} > \|T_j(u_j) - T_j(v_j)\|_{Y,q};$$

and $m \in I_2$ iff $\exists k_m \exists^\infty j \exists u_j, v_j \in F_{b_j}$ satisfying that $\frac{1}{k_m} \leq \|u_j - v_j\|_X \leq 1$ and

$$\|T_j(u_j) - T_j(v_j)\|_{Y,q} > 2^m \|u_j - v_j\|_X^{\frac{p}{q}}.$$

From the assumption, we can see that $I_1 \cup I_2 = \mathbb{N}$. Now we consider the following two cases.

Case 1. $|I_1| = \infty$. Select a finite set $J_1^m \subseteq \mathbb{N}$ for every $m \in I_1$ satisfying that

(i) $|J_1^m| \leq k_m^p$;
(ii) $1 \leq \sum_{j \in J_1^m} \|u_j - v_j\|_X^p \leq 2$;
(iii) if $m_1 < m_2$, then $\max J_1^{m_1} < \min J_1^{m_2}$.

Now we define $x, y \in \prod_{j\in\mathbb{N}} Z_{b_j}$ by

$$\begin{cases} x(j) = u_j, y(j) = v_j, & j \in J_1^m, m \in I_1, \\ x(j) = y(j) = 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\sum_{j\in\mathbb{N}} \|x(j) - y(j)\|_X^p = \sum_{m\in I_1} \sum_{j \in J_1^m} \|u_j - v_j\|_X^p \geq \sum_{m\in I_1} 1 = +\infty,$$

so $(x, y) \notin E((Z_{b_j})_{j\in\mathbb{N}}, p)$. One the other hand, we have

$$\begin{aligned} \textstyle\sum_{j\in\mathbb{N}} \|T_j(x(j)) - T_j(y(j))\|_{Y,q}^q &= \textstyle\sum_{m\in I_1} \sum_{j\in J_1^m} \|T_j(u_j) - T_j(v_j)\|_{Y,q}^q \\ &< \textstyle\sum_{m\in I_1} \sum_{j\in J_1^m} \frac{1}{2^{mq}} \|u_j - v_j\|_X^p \\ &\leq 2 \textstyle\sum_{m\in I_1} \left(\frac{1}{2^q}\right)^m \\ &< +\infty, \end{aligned}$$

contradicting Lemma 4.5!

Case 2. $|I_2| = \infty$. Select a finite set $J_2^m \subseteq \mathbb{N}$ for every $m \in I_2$ satisfying that

(i) $|J_2^m| \leq k_m^p$;

(ii) $1 \le \sum_{j\in J_2^m} \|u_j - v_j\|_X^p \le 2$;

(iii) if $m_1 < m_2$, then $\max J_2^{m_1} < \min J_2^{m_2}$;

(iv) for $j \in J_2^m$, we have $m \le b_j$.

For $m \in I_2, j \in J_2^m$, since $m \le b_j$, by the definition of Z_{b_j} we have

$$u_j^i \overset{\text{Def}}{=} u_j + \frac{i}{2^m}(v_j - u_j) \in Z_{b_j} \quad (i = 0, 1, \cdots, 2^m).$$

The triangle inequality gives

$$\sum_{1\le i\le 2^m} \|T_j(u_j^{i-1}) - T_j(u_j^i)\|_{Y,q} \ge \|T_j(u_j) - T_j(v_j)\|_{Y,q},$$

thus there is an $i(j)$ such that

$$\|T_j(u_j^{i(j)-1}) - T_j(u_j^{i(j)})\|_{Y,q} \ge \frac{1}{2^m}\|T_j(u_j) - T_j(v_j)\|_{Y,q}.$$

Now we define $x, y \in \prod_{j\in\mathbb{N}} Z_{b_j}$ by

$$\begin{cases} x(j) = u_j^{i(j)-1}, y(j) = u_j^{i(j)}, & j \in J_2^m, m \in I_2, \\ x(j) = y(j) = 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} \sum_{j\in\mathbb{N}} \|x(j) - y(j)\|_X^p &= \sum_{m\in I_2} \sum_{j\in J_2^m} \|u_j^{i(j)-1} - u_j^{i(j)}\|_X^p \\ &= \sum_{m\in I_2} \sum_{j\in J_2^m} \tfrac{1}{2^{mp}} \|u_j - v_j\|_X^p \\ &\le 2 \sum_{m\in I_2} (\tfrac{1}{2^p})^m \\ &< +\infty, \end{aligned}$$

so $(x, y) \in E((Z_{b_j})_{j\in\mathbb{N}}, p)$. One the other hand, we have

$$\begin{aligned} \sum_{j\in\mathbb{N}} \|T_j(x(j)) - T_j(y(j))\|_{Y,q}^q &= \sum_{m\in I_2} \sum_{j\in J_2^m} \|T_j(u_j^{i(j)-1}) - T_j(u_j^{i(j)})\|_{Y,q}^q \\ &= \sum_{m\in I_2} \sum_{j\in J_2^m} \big(\tfrac{1}{2^m}\|T_j(u_j) - T_j(v_j)\|_{Y,q}\big)^q \\ &> \sum_{m\in I_2} \sum_{j\in J_2^m} \|u_j - v_j\|_X^p \\ &\ge \sum_{m\in I_2} 1 \\ &= +\infty, \end{aligned}$$

contradicting Lemma 4.5 again! $\square$

Definition 4.7. *For two metric spaces $(M,d),(M',d')$ and $\alpha>0$. We say that M finitely Hölder(α) embeds into M' if there exists $A>0$ such that for every finite subset $F\subseteq M$, there is $T_F:F\to M'$ satisfying*

$$\frac{1}{A}d(u,v)^{\alpha}\le d'(T_F(u),T_F(v))\le Ad(u,v)^{\alpha}$$

for $u,v\in F$. While $\alpha=1$, we also say M finitely Lipschitz embeds into M'.

Theorem 4.8. *Let X,Y be two separable Banach spaces, $p,q\in[1,+\infty)$. The the following conditions are equivalent:*

(1) X finitely Hölder$(\frac{p}{q})$ embeds into $\ell_q(Y)$.

(2) For any sequence of finite subsets $(F_n)_{n\in\mathbb{N}}$ of X, we have

$$E((F_n)_{n\in\mathbb{N}},p)\le_B E(Y,q).$$

Proof. (1)⇒(2). Since X finitely Hölder$(\frac{p}{q})$ embeds into Y, we can find $A>0$, $T_n:F_n\to\ell_q(Y)$ such that

$$\frac{1}{A}\|u-v\|_X^{\frac{p}{q}}\le\|T_n(u)-T_n(v)\|_{Y,q}\le A\|u-v\|_X^{\frac{p}{q}}$$

for $u,v\in F_n$. Then $E((F_n)_{n\in\mathbb{N}},p)\le_B E(Y,q)$ follows from Theorem 4.2.

(2)⇒(1). Fix a sequence of finite subsets $F_n\subseteq X,n\in\mathbb{N}$ such that

$$\{0\}\subseteq F_0\subseteq F_1\subseteq\cdots\subseteq F_n\subseteq\cdots$$

and $\bigcup_{n\in\mathbb{N}}F_n$ is dense in X. Let $(b_j)_{j\in\mathbb{N}},(l_j)_{j\in\mathbb{N}}$ and $T_j:F_{b_j}\to Y^{l_{j+1}-l_j}$ be from Lemma 4.6. For convenience, we identify $(Y^{l_{j+1}-l_j},\|\cdot\|_{Y,q})$ with a subspace of $\ell_q(Y)$. Then T_j becomes a map $F_{b_j}\to\ell_q(Y)$.

Let us consider an arbitrary finite subset $F\subseteq X$. We can find two integers $c,d>0$ such that

$$\frac{1}{c}\le\|u-v\|_X\le d,$$

or equivalently,

$$\frac{1}{cd}\le\left\|\frac{u}{d}-\frac{v}{d}\right\|_X\le 1$$

for any distinct $u,v\in F$.

For every $u \in F$, since $\bigcup_{j\in\mathbb{N}} F_{b_j}$ is dense in X, there exists an $R(u) \in \bigcup_{j\in\mathbb{N}} F_{b_j}$ such that

$$\left\|\frac{u}{d} - R(u)\right\|_X < \frac{1}{4cd}.$$

Then for any distinct $u, v \in F$, we have

$$d\|R(u) - R(v)\|_X < d\left(\left\|\frac{u}{d} - \frac{v}{d}\right\|_X + \frac{1}{2cd}\right) = \|u - v\|_X + \frac{1}{2c} \leq 2\|u - v\|_X,$$

and

$$d\|R(u) - R(v)\|_X > d\left(\left\|\frac{u}{d} - \frac{v}{d}\right\|_X - \frac{1}{2cd}\right) = \|u - v\|_X - \frac{1}{2c} \geq \frac{1}{2}\|u - v\|_X.$$

From Lemma 4.6, there exist $m \in \mathbb{N}$ and a sufficiently large i such that

(i) $R(u) \in F_{b_i}$ for every $u \in F$;

(ii) for $u, v \in F_{b_i}$, if $\frac{1}{cd} \leq \|u - v\|_X \leq 1$, then

$$\frac{1}{2^m}\|u - v\|_X^{\frac{p}{q}} \leq \|T_i(u) - T_i(v)\|_{Y,q} \leq 2^m\|u - v\|_X^{\frac{p}{q}}.$$

We define $T_F : F \to \ell_q(Y)$ by

$$T_F(u) = d^{\frac{p}{q}} T_i(R(u))$$

for $u \in F$. Then for any distinct $u, v \in F$ we have

$$\begin{aligned} \|T_F(u) - T_F(u)\|_{Y,q} &= d^{\frac{p}{q}}\|T_i(R(u)) - T_i(R(v))\|_{Y,q} \\ &\leq 2^m(d\|R(u) - R(v)\|_X)^{\frac{p}{q}} \\ &< 2^{m+\frac{p}{q}}\|u - v\|_X^{\frac{p}{q}}, \end{aligned}$$

and

$$\begin{aligned} \|T_F(u) - T_F(u)\|_{Y,q} &= d^{\frac{p}{q}}\|T_i(R(u)) - T_i(R(v))\|_{Y,q} \\ &\geq 2^{-m}(d\|R(u) - R(v)\|_X)^{\frac{p}{q}} \\ &> 2^{-(m+\frac{p}{q})}\|u - v\|_X^{\frac{p}{q}}. \end{aligned}$$

Thus $A = 2^{m+\frac{p}{q}}$ witness that X finitely Hölder($\frac{p}{q}$) embeds into $\ell_q(Y)$. $\square$

5. Finitely Hölder(α) embeddability between Banach spaces

It is not surprising that finitely Hölder(α) embeddability is related to ultraproducts of Banach spaces. An ultrafilter $\mathfrak{A}$ on $\mathbb{N}$ is called free if it does not contain any finite set. Let U be a Banach space. Consider the space $\ell_\infty(U)$ of all bounded sequences $x \in U^{\mathbb{N}}$ with the norm $\|x\| = \sup_{n\in\mathbb{N}} \|x(n)\|_U$. Its subspace $N = \{x : \lim_{\mathfrak{A}} \|x(n)\|_U = 0\}$ is closed. The ultraproduct $(U)_{\mathfrak{A}}$ is the quotient space $\ell_\infty(U)/N$ with the norm $\|(x)_{\mathfrak{A}}\|_{\mathfrak{A}} = \lim_{\mathfrak{A}} \|x(n)\|_U$. For more details on ultraproducts in Banach space theory, see [9].

Theorem 5.1. *Let X, U be two Banach spaces, $\alpha > 0$, and let $\mathfrak{A}$ be a free ultrafilter on $\mathbb{N}$. Then X finitely Hölder(α) embeds into U iff X Hölder(α) embeds into $(U)_{\mathfrak{A}}$.*

Proof. ($\Rightarrow$). Fix a sequence of finite subsets $F_n \subseteq X, n \in \mathbb{N}$ such that

$$\{0\} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

and $\bigcup_{n\in\mathbb{N}} F_n$ is dense in X. There are $A > 0$ and $T_n : F_n \to U$ such that

$$\frac{1}{A}\|u - v\|_X^\alpha \leq \|T_n(u) - T_n(v)\|_U \leq A\|u - v\|_X^\alpha$$

for $u, v \in F_n$. Let $u \in \bigcup_{n\in\mathbb{N}} F_n$, $m = \min\{n : u \in F_n\}$, we define

$$T(u) = (0, \cdots, 0, T_m(u), T_{m+1}(u), \cdots)_{\mathfrak{A}}.$$

By the definition of the norm on $(U)_{\mathfrak{A}}$, it is easy to check that

$$\frac{1}{A}\|u - v\|_X^\alpha \leq \|T(u) - T(v)\|_{\mathfrak{A}} \leq A\|u - v\|_X^\alpha$$

for $u, v \in \bigcup_{n\in\mathbb{N}} F_n$. Since $\bigcup_{n\in\mathbb{N}} F_n$ is dense in X, we can extend T onto X.

($\Leftarrow$). Let $T : X \to (U)_{\mathfrak{A}}$ be a Hölder(α) embedding with the constant $A > 0$.

Fix a finite subset $F \subseteq X$. For $u, v \in F$, since

$$\frac{1}{A}\|u - v\|_X^\alpha \leq \|T(u) - T(v)\|_{\mathfrak{A}} = \lim_{\mathfrak{A}} \|T(u)(n) - T(v)(n)\|_U \leq A\|u - v\|_X^\alpha,$$

we have

$$I_{u,v} \overset{Def}{=} \left\{n : \frac{1}{A+1}\|u - v\|_X^\alpha \leq \|T(u)(n) - T(v)(n)\|_U \leq (A+1)\|u - v\|_X^\alpha\right\} \in \mathfrak{A}.$$

Now we fix an $m \in \bigcap_{u,v\in F} I_{u,v}$. For $u \in F$, we define $T_F(u) = T(u)(m)$ as desired. $\square$

By using this theorem, we can transfer the problem of finitely Lipschitz embeddability to the existence of Lipschitz embeddings. The latter was deeply studied in geometric nonlinear functional analysis. But this method does not work while $\alpha \neq 1$, because there is no more known result on the existence of Hölder(α) embeddings. Most recently, we employed other powerful tools, i.e., metric type and metric cotype, to solve this problem.

Lemma 5.2. *Let X, U be two Banach spaces, $\alpha > 0$. If X finitely Hölder(α) embeds into U, then $\alpha \leq 1$.*

Proof. Fix an $e \in X$ such that $\|e\|_X = 1$. Denote $F_n = \left\{\frac{i}{n}e : 0 \leq i \leq n\right\}$. There exist $A > 0$ and $T_n : F_n \to U$ such that

$$\frac{1}{A} = \frac{1}{A}\|e - 0\|_X^\alpha \leq \|T_n(e) - T_n(0)\|_U \leq A\|e - 0\|_X^\alpha,$$

and for $1 \leq i \leq n$,

$$\frac{1}{A}\left\|\frac{i}{n}e - \frac{i-1}{n}e\right\|_X^\alpha \leq \left\|T_n\left(\frac{i}{n}e\right) - T_n\left(\frac{i-1}{n}e\right)\right\|_U \leq A\left\|\frac{i}{n}e - \frac{i-1}{n}e\right\|_X^\alpha = \frac{A}{n^\alpha}.$$

The triangle inequality gives

$$\|T_n(e) - T_n(0)\|_U \leq \sum_{i=1}^{n}\left\|T_n\left(\frac{i}{n}e\right) - T_n\left(\frac{i-1}{n}e\right)\right\|_U.$$

Thus $\frac{1}{A} \leq n \cdot \frac{A}{n^\alpha}$, i.e.,

$$\frac{1}{A^2} \leq n^{1-\alpha},$$

Letting $n \to \infty$, yields $\alpha \leq 1$. $\square$

The notion of metric type was introduced in [4]. Let (M, d) be a metric space. For a map $H : \{0,1\}^n \to M$ and $s, s' \in \{0,1\}^n$, an unordered pair $\{H(s), H(s')\}$ is called an *edge* if s and s' are different at exactly one coordinate; and it is called a *diagonal* if s and s' are different at all n coordinates. Denote by E the set of all edges and by D the set of all diagonals. Clearly, $|E| = n2^{n-1}, |D| = 2^{n-1}$.

Definition 5.3 (J. Bougain, V. Milman, H. Wolfson). *Let $p \geq 1$. A metric space (M, d) has metric type p if there is a constant C, such that for every n and any $H : \{0,1\}^n \to M$ the following inequality holds:*

$$\left(\sum_D d(H(s), H(s'))^2\right)^{\frac{1}{2}} \leq C n^{\frac{1}{p}-\frac{1}{2}} \left(\sum_E d(H(s), H(s'))^2\right)^{\frac{1}{2}}.$$

where the sums range over all the diagonals and all the edges respectively.

Theorem 5.4. *Let X, U be two infinite dimensional Banach spaces, $\alpha > 0$. If X finitely Hölder(α) embeds into U, then*

$$\frac{p(X)}{p(U)} \geq \alpha.$$

Proof. The Maurey-Pisier Theorem [13] says that X contains $\ell^n_{p(X)}$'s uniformly. Thus for every $n \in \mathbb{N}$, there is a linear operator $L_n : \ell^n_{p(X)} \to X$ such that for all $s \in \ell^n_{p(X)}$ we have $\|s\|_{p(X)} \leq \|L_n(s)\|_X \leq 2\|s\|_{p(X)}$.

Note that $\{0,1\}^n \subseteq \ell^n_{p(X)}$, we denote $F_n = L_n(\{0,1\}^n)$. There are $A > 0$ and $T_n : F_n \to U$ such that for $u, v \in F_n$ we have

$$\frac{1}{A}\|u - v\|_X^\alpha \leq \|T_n(u) - T_n(v)\|_U \leq A\|u - v\|_X^\alpha.$$

Fix a $p < p(U)$. By [4], Corollary 5.10, U has metric type p. Denote $H = T_n \circ L_n$. Now we estimate lengths of edges and diagonals for H as follows.

Let $s, s' \in \{0,1\}^n$. If s and s' are different at exactly one coordinate, then $\|s - s'\|_{p(X)} = 1$, $\|L_n(s) - L_n(s')\|_X \leq 2$. So

$$\|H(s) - H(s')\|_U = \|T_n(L_n(s)) - T_n(L_n(s'))\|_U \leq A2^\alpha.$$

On the other hand, if s and s' are different at all n coordinates, then $\|s - s'\|_{p(X)} = n^{\frac{1}{p(X)}}$, $\|L_n(s) - L_n(s')\|_X \geq n^{\frac{1}{p(X)}}$. So

$$\|H(s) - H(s')\|_U = \|T_n(L_n(s)) - T_n(L_n(s'))\|_U \geq \frac{n^{\frac{\alpha}{p(X)}}}{A}.$$

Therefore,

$$\begin{aligned} 2^{n-1} n^{\frac{2\alpha}{p(X)}} A^{-2} &\leq \textstyle\sum_D \|H(s) - H(s')\|_U^2 \\ &\leq C^2 n^{2(\frac{1}{p}-\frac{1}{2})} \textstyle\sum_E \|H(s) - H(s')\|_U^2 \\ &\leq C^2 n^{2(\frac{1}{p}-\frac{1}{2})} n 2^{n-1} A^2 2^{2\alpha}. \end{aligned}$$

Thus

$$n^{\frac{\alpha}{p(X)}-\frac{1}{p}} \leq CA^2 2^\alpha.$$

By letting $n \to \infty$, we see that $\frac{\alpha}{p(X)} \leq \frac{1}{p}$ for every $p < p(U)$. It then follows that $\frac{\alpha}{p(X)} \leq \frac{1}{p(U)}$, i.e., $\frac{p(X)}{p(U)} \geq \alpha$. $\square$

The notion of metric cotype introduced by M. Mendel and A. Naor [14] is more complicated than metric type.

Definition 5.5 (M. Mendel, A. Naor). *Let $q > 0$. A metric space (M, d) has metric cotype q if there is a constant Γ, which satisfies that for every n, there exists an even integer m, such that for every $H : \mathbb{Z}_m^n \to M$, the following inequality holds:*

$$\sum_{j=1}^{n} \mathbb{E}_s \left[d\left(H\left(s + \frac{m}{2}e_j\right), H(s)\right)^q \right] \leq \Gamma^q m^q \mathbb{E}_{\epsilon,s}[d(H(s+\epsilon), H(s))^q],$$

where the expectations $\mathbb{E}_s$ and $\mathbb{E}_{\epsilon,s}$ above are taken with respect to uniformly chosen $s \in \mathbb{Z}_m^n$ and $\epsilon \in \{-1,0,1\}^n$, and $e_j = (0,\cdots,0,\overset{j}{1},0,\cdots,0)$ for $j = 1,\cdots,n$.

M. Mendel and A. Naor showed that, for a Banach space, metric cotype is coincide with cotype (see [14], Theorem 1.2). They also estimated, in Theorem 4.1 of [14], the minimal m in Definition 5.5 for K-convex Banach spaces.

Lemma 5.6 (M. Mendel, A. Naor). *Let X be a K-convex Banach space with cotype q. Then there exists constant $C > 0$ such that for every n and every integer $m \geq Cn^{\frac{1}{q}}$ which is divisible by 4, the inequality in Definition 5.5 holds.*

Theorem 5.7. *Let X, U be two infinite dimensional Banach spaces with $p(U) > 1$. For $\alpha > 0$, if X finitely Hölder(α) embeds into U, then*

$$q(X) \leq q(U).$$

Proof. For every n, m, the map $\sigma_n : \mathbb{Z}_m^n \to \mathbb{C}^n$ is defined by

$$\sigma_n(k_1,\cdots,k_n) = \left(\exp\left(\frac{2\pi k_1}{m}i\right),\cdots,\exp\left(\frac{2\pi k_n}{m}i\right)\right)$$

for $k_1, \cdots, k_n \in \mathbb{Z}_m$. Note that $\ell^n_{q(X)}(\mathbb{C}) \cong \ell^n_{q(X)}(\mathbb{R}^2)$, so it is L-isomorphic to $\ell^{2n}_{q(X)}$ where $L > 0$ is a constant independent to n. By the Maurey-Pisier Theorem [13], X contains $\ell^n_{q(X)}$'s uniformly. Thus we can find a linear operation $R_n : \ell^n_{q(X)}(\mathbb{C}) \to X$ and $P, Q > 0$ independent of n such that

$$P\|s\|_{\mathbb{C},q(X)} \leq \|R_n(s)\|_X \leq Q\|s\|_{\mathbb{C},q(X)}$$

for $s \in \ell^n_{q(X)}(\mathbb{C})$.

We denote $F_n = R_n(\sigma_n(\mathbb{Z}^n_m))$. There are $A > 0$ and $T_n : F_n \to U$ such that for $u, v \in F_n$ we have

$$\frac{1}{A}\|u-v\|^\alpha_X \leq \|T_n(u) - T_n(v)\|_U \leq A\|u-v\|^\alpha_X.$$

Fix a $q > q(U)$. Denote $H = T_n \circ R_n \circ \sigma_n$. By the Pisier's K-convexity theorem [15], $p(U) > 1$ iff U is K-convex. Since U has cotype q, from Lemma 5.6, there is a constant $C > 0$ and for every sufficiently large n, we can find a suitable m such that m is divisible by 4, $Cn^{\frac{1}{q}} \leq m \leq (C+1)n^{\frac{1}{q}}$ and

$$\sum_{j=1}^n \mathbb{E}_s\left[\left\|H\left(s + \frac{m}{2}e_j\right) - H(s)\right\|^q_U\right] \leq \Gamma^q m^q \mathbb{E}_{\epsilon,s}[\|H(s+\epsilon) - H(s)\|^q_U].$$

For $s \in \mathbb{Z}^n_m$ and $\epsilon \in \{-1,0,1\}^n$, we have

$$\begin{aligned}\|\sigma_n(s+\epsilon) - \sigma_n(s)\|_{\mathbb{C},q(X)} &= \left(\sum_{j=1}^n \left|\exp\left(\tfrac{2\pi(s_j+\epsilon_j)}{m}i\right) - \exp\left(\tfrac{2\pi s_j}{m}i\right)\right|^{q(X)}\right)^{\frac{1}{q(X)}} \\ &\leq \left(\sum_{j=1}^n \left|\tfrac{2\pi\epsilon_j}{m}\right|^{q(X)}\right)^{\frac{1}{q(X)}} \leq \tfrac{2\pi}{m} n^{\frac{1}{q(X)}},\end{aligned}$$

so

$$\begin{aligned}\|H(s+\epsilon) - H(s)\|_U &\leq A\|R_n(\sigma_n(s+\epsilon)) - R_n(\sigma_n(s))\|^\alpha_X \\ &\leq AQ^\alpha \|\sigma_n(s+\epsilon) - \sigma_n(s)\|^\alpha_{\mathbb{C},q(X)} \\ &\leq AQ^\alpha (2\pi)^\alpha n^{\frac{\alpha}{q(X)}} m^{-\alpha}.\end{aligned}$$

Moreover, for $j = 1, 2, \cdots, n$, we have

$$\left\|\sigma_n\left(s + \frac{m}{2}e_j\right) - \sigma_n(s)\right\|_{\mathbb{C},q(X)} = \left|\exp\left(\frac{2\pi s_j}{m}i + \pi i\right) - \exp\left(\frac{2\pi s_j}{m}i\right)\right| = 2,$$

so

$$\begin{aligned}\left\|H\left(s+\tfrac{m}{2}e_j\right)-H(s)\right\|_U &\geq \tfrac{1}{A}\left\|R_n\left(\sigma_n\left(s+\tfrac{m}{2}e_j\right)\right)-R_n(\sigma_n(s))\right\|_X^\alpha \\ &\geq A^{-1}P^\alpha\left\|\sigma_n\left(s+\tfrac{m}{2}e_j\right)-\sigma_n(s)\right\|_{\mathbb{C},q(X)}^\alpha \\ &\geq 2^\alpha A^{-1}P^\alpha.\end{aligned}$$

Therefore,

$$\begin{aligned}n(2^\alpha A^{-1}P^\alpha)^q &\leq \textstyle\sum_{j=1}^n \mathbb{E}_s\left[\left\|H\left(s+\tfrac{m}{2}e_j\right)-H(s)\right\|_U^q\right] \\ &\leq \Gamma^q m^q(AQ^\alpha(2\pi)^\alpha n^{\frac{\alpha}{q(X)}}m^{-\alpha})^q,\end{aligned}$$

i.e.,

$$n^{\frac{1}{q}-\frac{\alpha}{q(X)}}m^{\alpha-1}\leq W\stackrel{Def}{=}\Gamma A^2P^{-\alpha}Q^\alpha\pi^\alpha.$$

Lemma 5.2 gives $\alpha\leq 1$. Since $m\leq (C+1)n^{\frac{1}{q}}$ for sufficiently large n, we have $m^{\alpha-1}\geq (C+1)^{\alpha-1}n^{\frac{\alpha-1}{q}}$. Thus

$$n^{\frac{1}{q}-\frac{1}{q(X)}}\leq (W(C+1)^{-(\alpha-1)})^{\frac{1}{\alpha}}.$$

By letting $n\to\infty$, we see that $\frac{1}{q}\leq\frac{1}{q(X)}$ for every $q>q(U)$. It then follows that $q(X)\leq q(U)$. □

6. Applications to classical Banach spaces

In this section, we compare equivalence relations $E(X,p)$'s, where $p\in\{0\}\cup[1,+\infty)$ and X is one of classical Banach spaces, namely, $X=c_0, C[0,1]$ or $X=\ell_r, L_r$ for $r\in[1,+\infty)$.

Firstly, we present all reducibility concerning the case $p=0$.

Theorem 6.1. *Let X be a separable Banach space, $p\in[1,+\infty)$. Then $E(X,0)\sim_B\mathbb{R}^{\mathbb{N}}/c_0$, and $E(X,p)$ is not Borel comparable with $\mathbb{R}^{\mathbb{N}}/c_0$.*

Proof. $\mathbb{R}^{\mathbb{N}}/c_0\leq_B E(X,0)$ is trivial, and $E(X,0)\leq_B\mathbb{R}^{\mathbb{N}}/c_0$ follows from Theorem 3.4.

R. Dougherty and G. Hjorth showed that $\mathbb{R}^{\mathbb{N}}/\ell_1\not\leq_B\mathbb{R}^{\mathbb{N}}/c_0$ and $\mathbb{R}^{\mathbb{N}}/\ell_1\leq_B\mathbb{R}^{\mathbb{N}}/\ell_p$ (see [10], Theorem 6.1, and [5], Theorem 1.1). By Corollary 4.3, we have $\mathbb{R}^{\mathbb{N}}/\ell_p=E(\mathbb{R},p)\leq_B E(X,p)$. Therefore, $E(X,p)\not\leq_B\mathbb{R}^{\mathbb{N}}/c_0$.

Suppose $\mathbb{R}^{\mathbb{N}}/c_0\leq E(X,p)$. Denote $Z_n=I_n=\left\{\frac{k}{2^n}:k\in\mathbb{N},0\leq k\leq 2^n\right\}$. By Lemma 4.5, there exist strictly increasing sequences of natural numbers

$(b_j)_{j\in\mathbb{N}}, (l_j)_{j\in\mathbb{N}}$ and $T_j : Z_{b_j} \to X^{l_{j+1}-l_j}$ such that, for $x, y \in \prod_{j\in\mathbb{N}} Z_{b_j}$, we have

$$(x,y) \in \mathbb{R}^{\mathbb{N}}/c_0 \iff \sum_{j\in\mathbb{N}} \|T_j(x(j)) - T_j(y(j))\|^p_{X,p} < +\infty.$$

By the same trick of the proof of [5], Theorem 2.2, Claim (iv), we obtain a contradiction. $\square$

Lemma 6.2. *For any $\alpha \in (0,1]$, there is an $n \in \mathbb{N}$ such that $\mathbb{R}$ Hölder(α) embeds into $\mathbb{R}^n$.*

Proof. It will suffice to prove for the case $\frac{1}{2} < \alpha < 1$.

Let $r = 4^{-\alpha}$. Then $\frac{1}{4} < r < \frac{1}{2}$. Proposition 1.2 of [5] gives a Hölder(α) embedding $K_r : [0,1] \to \mathbb{R}^2$. As remarked in [5], we can extend K_r to all of $\mathbb{R}$ as follows.

First step, we extend K_r to a Hölder(α) embedding $K_r^1 : [0,4] \to \mathbb{R}^2$ by

$$K_r^1(t) = r^{-1} K_r\left(\frac{t}{4}\right)$$

for $t \in [0,4]$. From the definition of $K_r(t)$, note that

$$K_r(0) = (0,0), K_r\left(\frac{1}{4}\right) = (r,0), K_r\left(\frac{3}{4}\right) = (1-r,0), K_r(1) = (1,0),$$

we can see that $K_r^1 \restriction [0,1] = K_r$.

Second step, we extend K_r^1 to a Hölder(α) embedding $K_r^2 : [-12,4] \to \mathbb{R}^2$ by

$$K_r^2(t) = r^{-1} K_r^1\left(\frac{t+12}{4}\right) - (r^{-2} - r^{-1}, 0)$$

for $t \in [-12,4]$. From the definition of $K_r(t)$, note that

$$K_r^1(0) = (0,0), K_r^1(3) = (r^{-1}-1,0), K_r^1(4) = (r^{-1},0),$$

we can see that $K_r^2 \restriction [0,4] = K_r^1$.

Repeating these steps, we can extend K_r to a Hölder(α) embedding $K_r^\infty : \mathbb{R} \to \mathbb{R}^2$.

For $\frac{1}{2^k} < \alpha < 1$, by repeatedly applying K_r^∞ for some suitable r, we can find a Hölder(α) embedding $\mathbb{R} \to \mathbb{R}^{2^k}$. $\square$

Theorem 6.3. *For $r, s, p, q \in [1, +\infty)$, we have*

(1) $E(\ell_r, p) \leq_B E(L_r, p)$*;*

(2) $E(\ell_p, p) \sim_B \mathbb{R}^{\mathbb{N}}/\ell_p \leq_B E(\ell_r, p)$*;*

(3) $E(\ell_2, p) \sim_B E(L_2, p) \leq_B E(L_r, p)$*;*

(4) if $s \leq r \leq 2$*, then* $E(L_r, p) \leq_B E(L_s, p)$*;*

(5) if $\frac{r}{p} = \frac{s}{q}, p \leq q$*, then* $E(\ell_r, p) \leq_B E(\ell_s, q)$ *and* $E(L_r, p) \leq_B E(L_s, q)$*.*

Proof. Let $X \hookrightarrow Y$ stand for that X Lipschitz embeds into Y. From Corollary 4.3, clauses (1)-(4) are given by following well known facts.

(1) $\ell_r \hookrightarrow L_r \hookrightarrow \ell_p(L_r)$.

(2) $\mathbb{R}^{\mathbb{N}}/\ell_p = E(\mathbb{R}, p)$ and $\mathbb{R} \hookrightarrow X$ for any Banach space.

(3) $\ell_2 \cong L_2$ and $L_2 \hookrightarrow L_r$ (see [3], pp. 189).

(4) if $s \leq r \leq 2$, then $L_r \hookrightarrow L_s$ (see [3], Corollary 8.8).

(5) Denote $\alpha = \frac{r}{s} = \frac{p}{q} \in (0, 1]$. By Lemma 6.2, there are $n \in \mathbb{N}$ and a Hölder(α) embedding $T : \mathbb{R} \to \mathbb{R}^n$. Hence, there exists $A > 0$, such that for $t_1, t_2 \in \mathbb{R}$, we have

$$\frac{1}{A}|t_1 - t_2|^\alpha \leq \|T(t_1) - T(t_2)\|_2 \leq A|t_1 - t_2|^\alpha.$$

Now we define a Hölder(α) embedding $\tilde{T} : L_r \to L_s$. For $f \in L_r$ and $t \in (0, 1]$, if $t \in \left(\frac{k-1}{n}, \frac{k}{n}\right]$ for some $k \in \{1, 2, \cdots, n\}$, let

$$\tilde{T}(f)(t) = n^{\frac{1}{s}} T(f(nt - k + 1))(k).$$

For $f, g \in L_r$, denote $\tau = nt - k + 1$, then

$$\int_{\frac{k-1}{n}}^{\frac{k}{n}} |\tilde{T}(f)(t) - \tilde{T}(g)(t)|^s dt = \int_0^1 |T(f(\tau))(k) - T(g(\tau))(k)|^s d\tau.$$

Therefore,

$$\begin{aligned} \int_0^1 |\tilde{T}(f)(t) - \tilde{T}(g)(t)|^s dt &= \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} |\tilde{T}(f)(t) - \tilde{T}(g)(t)|^s dt \\ &= \int_0^1 \sum_{k=1}^n |T(f(\tau))(k) - T(g(\tau))(k)|^s d\tau \\ &= \int_0^1 \|T(f(\tau)) - T(g(\tau))\|_s^s d\tau. \end{aligned}$$

Since

$$\frac{1}{n}\|u\|_2 \leq \|u\|_\infty \leq \|u\|_s \leq n\|u\|_\infty \leq n\|u\|_2,$$

for $u \in \mathbb{R}^n$, we have

$$\begin{aligned}\int_0^1 |\tilde{T}(f)(t) - \tilde{T}(g)(t)|^s dt \ &\leq n^s \int_0^1 \|T(f(\tau)) - T(g(\tau))\|_2^s d\tau \\ &\leq n^s A^s \int_0^1 |f(\tau) - g(\tau)|^r d\tau,\end{aligned}$$

and

$$\int_0^1 |\tilde{T}(f)(t) - \tilde{T}(g)(t)|^s dt \geq \frac{1}{n^s A^s} \int_0^1 |f(\tau) - g(\tau)|^r d\tau.$$

It follows that

$$\frac{1}{nA}\|f - g\|_r^\alpha \leq \|\tilde{T}(f) - \tilde{T}(g)\|_s \leq nA\|f - g\|_r^\alpha.$$

Thus $\tilde{T}$ is a Hölder(α) embedding. Since $L_s \hookrightarrow \ell_q(L_s)$ and $\alpha = \frac{p}{q}$, L_r Hölder($\frac{p}{q}$) embeds into $\ell_q(L_s)$. Then $E(L_r, p) \leq_B E(L_s, q)$ follows from Corollary 4.3.

Similarly, we can prove that $E(\ell_r, p) \leq_B E(\ell_s, q)$. $\square$

Theorem 6.4. *For $r, s, p, q \in [1, +\infty)$, if $E(\ell_r, p) \leq_B E(\ell_s, q)$ or $E(L_r, p) \leq_B E(L_s, q)$, then we have*

(1) $p \leq q$;
(2) $\min\left\{\frac{r}{p}, \frac{2}{p}\right\} \geq \min\left\{\frac{s}{q}, 1, \frac{2}{q}\right\}$;
(3) $\max\{r, 2\} \leq \max\{s, q, 2\}$.

Proof. Recall that

$$p(\ell_s) = p(L_s) = \min\{s, 2\}, \quad q(\ell_s) = q(L_s) = \max\{s, 2\},$$

and for any Banach space X,

$$p(\ell_q(X)) = \min\{p(X), q\}, \quad q(\ell_q(X)) = \max\{q(X), q\}.$$

Thus

$$\begin{aligned}p(\ell_q(\ell_s)) &= p(\ell_q(L_s)) = \min\{s, q, 2\}, \\ q(\ell_q(\ell_s)) &= q(\ell_q(L_s)) = \max\{s, q, 2\}.\end{aligned}$$

Therefore, clauses (1),(2) and the case $\min\{s, q\} > 1$ in clause (3) follow from Theorems 4.8,5.4,5.7 and Lemma 5.2.

For the case $\min\{s, q\} = 1$, let $\alpha \in (0, 1)$ be arbitrary. By Theorem 6.3, (5), we have

$$E(\ell_s, q) \leq_B E\left(\ell_{\frac{s}{\alpha}}, \frac{q}{\alpha}\right), \quad E(L_s, q) \leq_B E\left(L_{\frac{s}{\alpha}}, \frac{q}{\alpha}\right).$$

Thus $\max\{r, 2\} \leq \max\left\{\frac{s}{\alpha}, \frac{q}{\alpha}, 2\right\}$ for any $\alpha \in (0, 1)$. It follows that $\max\{r, 2\} \leq \max\{s, q, 2\}$. $\square$

Corollary 6.5. *For $r, s, p, q \in [1, +\infty)$, if $E(\ell_r, p) \sim_B E(\ell_s, q)$ or $E(L_r, p) \sim_B E(L_s, q)$, then we have $p = q$ and*

$$r = s \quad \text{or} \quad p \leq r, s \leq 2 \quad \text{or} \quad 2 \leq r, s \leq p.$$

Corollary 6.6. *For $r, s \in [1, 2]$ and $p, q \in [1, +\infty)$, if $s \leq q$, then*

$$E(L_r, p) \leq_B E(L_s, q) \iff p \leq q, \ \frac{r}{p} \geq \frac{s}{q}.$$

Proof. "$\Rightarrow$" follows from Theorem 6.4.

($\Leftarrow$). Assume that $2^k \leq \frac{qr}{p} \leq 2^{k+1}$ for some $k \in \mathbb{N}$. From Theorem 6.3, (4) and (5), we have

$$\begin{aligned} E(L_r, p) &\leq_B E\left(L_2, \tfrac{2p}{r}\right) \leq_B E\left(L_1, \tfrac{2p}{r}\right) \leq_B E\left(L_2, \tfrac{4p}{r}\right) \leq_B \cdots \\ &\leq_B E\left(L_2, \tfrac{2^j p}{r}\right) \leq_B E\left(L_1, \tfrac{2^j p}{r}\right) \leq_B E\left(L_2, \tfrac{2^{j+1} p}{r}\right) \leq_B \cdots \\ &\leq_B E\left(L_2, \tfrac{2^k p}{r}\right). \end{aligned}$$

If $s \geq \frac{qr}{2^k p}$, denote $s_1 = \frac{2^k ps}{qr}$. Then $s_1 \in [1, 2]$, we have

$$E\left(L_2, \frac{2^k p}{r}\right) \leq_B E\left(L_{s_1}, \frac{2^k p}{r}\right) \leq_B E(L_s, q).$$

Otherwise, denote $s_2 = \frac{qr}{2^k p}$. Then $s_2 \in [1, 2]$ and $s < s_2$, we have

$$E\left(L_2, \frac{2^k p}{r}\right) \leq_B E\left(L_1, \frac{2^k p}{r}\right) \leq_B E(L_{s_2}, q) \leq_B E(L_s, q).$$

$\square$

Corollary 6.7. *For $r, s, p, q \in [1, +\infty)$, if $s \le q \le 2$, then*

$$E(L_r, p) \le_B E(L_s, q) \iff p \le q,\ r \le 2,\ \frac{r}{p} \ge \frac{s}{q}.$$

Proof. "$\Rightarrow$" follows from Theorem 6.4, and "$\Leftarrow$" follows from Corollary 6.6. $\square$

In the end, we settle down the case $X = c_0$ or $C[0, 1]$.

Theorem 6.8. *For $r, p, q \in [1, +\infty)$, we have*

(1) $E(C[0, 1], p) \sim_B E(c_0, p)$;
(2) $E(L_r, p) \le_B E(c_0, p)$;
(3) $q \in [1, +\infty)$, $E(c_0, p) \not\le_B E(L_r, q)$;
(4) if $p < q$, then $E(c_0, p) <_B E(c_0, q)$.

Proof. $E(c_0, p) \le E(C[0, 1], p)$ is trivial. From Theorem 3.3, $C[0, 1] \hookrightarrow c_0$ and $L_r \hookrightarrow c_0$. So clauses (1) and (2) hold.

(3) Fix an $s > \max\{r, q, 2\}$. Theorem 6.4, (3) shows $E(L_s, p) \not\le_B E(L_r, q)$. So $E(c_0, p) \not\le_B E(L_r, q)$, since $E(L_s, p) \le_B E(c_0, p)$.

(4) From Lemma 6.2, there are $n \in \mathbb{N}$ and a Hölder($\frac{p}{q}$) embedding $T : \mathbb{R} \to \mathbb{R}^n$. Fix a bijection $\langle \cdot, \cdot \rangle_n : \mathbb{N} \times \{1, 2, \cdots, n\} \to \mathbb{N}$. We define $\hat{T} : c_0 \to c_0$ by

$$\hat{T}(x)(\langle k, m \rangle_n) = T(x(k))(m)$$

for $k \in \mathbb{N}$ and $m = 1, 2, \cdots, n$. It is easy to check that $\hat{T}$ is a Hölder($\frac{p}{q}$) embedding. It follows that $E(c_0, p) \le_B E(c_0, q)$.

On the other hand, $E(c_0, q) \not\le_B E(c_0, p)$ follows from Lemma 5.2. $\square$

7. Further remarks

Perhaps the most curious problem is how to compare equivalence relations $E(X, p)$ and $E(Y, p)$ when X and Y have same types and cotypes. Especially, if $r \ne 2$, does $E(L_r, p) \sim_B E(\ell_r, p)$? Though $L_r \not\hookrightarrow \ell_r$, we have the following lemma.

Lemma 7.1. *For $r \in [1, +\infty)$, L_r finitely Lipschitz embeds into ℓ_r.*

Proof. Let $F = \{f_1, \cdots, f_n\} \subseteq L_r$. Denote $\varepsilon = \min\{\|f_i - f_j\|_r : 1 \leq i < j \leq n\} > 0$. Find continuous functions $\varphi_1, \cdots, \varphi_n$ such that for each i, we have $\|f_i - \varphi_i\|_r < \frac{\varepsilon}{8}$. Since all $\varphi_i(t)$'s are uniformly continuous on $[0, 1]$, there exists a sufficiently large m such that, for $k < m, i = 1, \cdots, n$ and $t \in \left[\frac{k}{m}, \frac{k+1}{m}\right]$, we have

$$\left|\varphi_i(t) - \varphi_i\left(\frac{k}{m}\right)\right| < \frac{\varepsilon}{8}.$$

Thus for $i, j = 1, \cdots, n$, we have

$$\left|\|\varphi_i - \varphi_j\|_r - \left(\sum_{k=0}^{m-1} \frac{1}{m}\left|\varphi_i\left(\frac{k}{m}\right) - \varphi_j\left(\frac{k}{m}\right)\right|^r\right)^{\frac{1}{r}}\right| < \frac{\varepsilon}{4}.$$

Define $T_F : F \to \ell_r$ by

$$T_F(f_i) = m^{-\frac{1}{r}}\left(\varphi_i(0), \varphi_i\left(\frac{1}{m}\right), \cdots, \varphi_i\left(\frac{m-1}{m}\right), 0, 0, \cdots\right)$$

for $i = 1, \cdots, n$. Then we can check that, for $1 \leq i < j \leq n$,

$$\frac{1}{2}\|f_i - f_j\|_r \leq \|f_i - f_j\|_r - \frac{\varepsilon}{2} \leq \|T_F(f_i) - T_F(f_j)\|_r \leq \|f_i - f_j\|_r + \frac{\varepsilon}{2} \leq 2\|f_i - f_j\|_r,$$

as desired. □

Question 7.2. *For $r, p \in [1, +\infty)$, if $r \neq 2$, does $E(L_r, p) \leq_B E(\ell_r, p)$?*

If the statement in this question is true, it will follow that, for $1 \leq p \leq r \leq 2$,

$$E(\ell_r, p) \leq_B E(L_r, p) \leq_B E(L_p, p) \leq_B E(\ell_p, p) \leq_B E(\ell_r, p).$$

Then we shall have

$$\mathbb{R}^{\mathbb{N}}/\ell_p \sim_B E(\ell_p, p) \sim_B E(\ell_r, p) \sim_B E(L_r, p) \sim_B E(L_2, p).$$

Though Corollary 6.6 gives an almost complete picture on Borel reducibility between $E(L_r, p)$'s for $r \in [1, 2]$, we know little about the case $r \geq 2$. So another problem is, whether clauses (1)-(3) in Theorem 6.4 can be a sufficient condition for $E(L_r, p) \leq_B E(L_s, q)$. This problem leads to the following question.

Question 7.3. *(1) For $r, s \geq 2$, if $r \leq s$, does L_r finitely Lipschitz embed into L_s? Furthermore, does $E(L_r, p) \leq_B E(L_s, p)$?*

(2) For $p, q \in [1, +\infty), r \geq 2$, if $p \leq q$, does L_r finitely Hölder($\frac{p}{q}$) embed into L_r itself? Furthermore, does $E(L_r, p) \leq_B E(L_r, q)$?

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